

## ON RAMANUJAN MASTER THEOREM

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ABSTRACT. In this short note we use the umbral formalism to derive the Ramanujan Master Theorem and discuss its extension to more general cases.

The Ramanujan Master Theorem (RMT) [1] states that given a function  $f(x)$  that in a neighborhood of the origin admits a series expansion of the form

$$(1) \quad f(x) = \sum_{n=0}^{\infty} \varphi(n) \frac{(-1)^n}{n!} x^n,$$

with  $\varphi(0) \neq 0$ <sup>1</sup>, then

$$(2) \quad \int_0^{\infty} dx x^{\nu-1} f(x) = \Gamma(\nu) \varphi(-\nu), \quad (\nu \in \mathbb{R}).$$

In a recent paper [2], the usefulness of the RMT for the evaluation of integrals associated to some Feynman diagrams has been emphasized.

The use of the formalism of the umbral calculus [3]

$$(3) \quad \varphi(n) = \hat{c}^n \varphi(0)$$

allows to cast eq.(1) in the form

$$(4) \quad f(x) = e^{-\hat{c}x} \varphi(0),$$

and, since eq.(3) holds  $\forall n \in \mathbb{R}$ , we formally get

$$(5) \quad \int_0^{\infty} dx x^{\nu-1} e^{-\hat{c}x} \varphi(0) = \Gamma(\nu) \hat{c}^{-\nu} \varphi(0) = \Gamma(\nu) \varphi(-\nu).$$

The procedure we have followed is not a rigorous proof but just an operational “tool” useful to formulate the RMT. We follow therefore the fairly pragmatic criterion of exploiting the umbral method to get a generalization of the RMT integration formula and then check it a posteriori for a number of specific examples.

Let us consider the case of a function  $f(x)$  that admits the following series expansion around the origin ( $m \in \mathbb{N}$ )

$$(6) \quad f(x) = \sum_{n=0}^{\infty} \varphi(n) \frac{(-1)^n}{n!} (x^m)^n$$

with  $f(0) = \varphi(0) \neq 0$ . Along the same lines followed for proving the identity (5), it is easy to show that

$$(7) \quad \begin{aligned} \int_0^{\infty} dx x^{\nu-k} f(x) &= \frac{1}{m} \Gamma\left(\frac{\nu+1-k}{m}\right) \hat{c}^{-\frac{\nu+1-k}{m}} \varphi(0) \\ &= \frac{1}{m} \Gamma\left(\frac{\nu+1-k}{m}\right) \varphi\left(-\frac{\nu+1-k}{m}\right), \end{aligned}$$

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<sup>1</sup>For  $\nu > 0$ , this condition guarantees the convergence of the integral near  $x = 0$ .

where, in this case, the convergence near to  $x = 0$  requires  $\nu > k - 1$ .

This result is clearly a conjecture, waiting for a more solid justification, but it has been checked numerically for different values of integers  $m$  and  $k$ , and for different types of functions  $f(x)$ . As an example of application of eq.(7), in the case  $\nu = k$ ,  $m = 2$  we get

$$(8) \quad \int_{-\infty}^{\infty} dx f(x) = \sqrt{\pi} \hat{c}^{-1/2} \varphi(0) = \sqrt{\pi} \varphi(-1/2).$$

Further discussions will be presented elsewhere in a more detailed investigation.

#### REFERENCES

- [1] see for example, H. M. Edwards, *Riemann's Zeta Function*, pp. 218-225, Dover, New York (2001).
- [2] I. González, V. H. Moll, I. Schmidt, [arXiv:1103.0588v1\[math-ph\]](#).
- [3] S. Roman, *The Umbral Calculus*, Dover Publications, New York (2005).

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